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UNICITY OF FAIR PARETO OPTIMAL RISK EXCHANGES

by
HANS BÜHLMANN
and
WILLIAM S. JEWELL

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UNICITY OF FAIR PARETO OPTIMAL RISK EXCHANGES[†]

by

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[††]This research was performed while visiting the University of California, Berkeley.

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ABSTRACT

In a related paper, the concept of a Fair Pareto-Optimal Risk Exchange is defined. Here the proofs for these results are summarized.

Unicity of Fair Pareto Optimal Risk Exchanges

by

Hans Bühlmann[1], Zürich, and William S. Jewell, Berkeley

0. Motivation

Risk exchanges are the most important instruments for improving the solvability of an insurance company without requiring additional capital. There are two aspects to such exchanges:

i) economical

ii) actuarial

The economic aspect leads to the condition of Pareto optimality, the actuarial to the condition of fairness. It is very remarkable, that if both conditions are imposed, the solution to the exchange problem becomes uniquely determined under rather weak technical assumptions.

1. Introduction

This paper gives the proofs for the three basic theorems in a forthcoming paper on risk pools [5]. Theorem 1 is usually referred to as Borch's [1] theorem in actuarial literature. Recently Gerber [2] has extended it for the case of effective boundary conditions. Our proof for this theorem is different as it shows the connection with the finite dimensional Kuhn-Tucker theorem. Theorem 2 is essentially due to Gale [3,4]. He states it in a different terminology and wants to find optimal fair distribution for economic goods which are liked. Our context and assumptions being different, we feel it necessary to give a new proof.

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[1] This research was performed while visiting the University of California, Berkeley.

2. Definitions

All our random variables are defined on a probability space $(\Omega, \mathfrak{A}, \mu)$. We then have a given random variable $X \geq 0$ which we interprete as total claims (or more generally total pooled payments).

Def. $\underline{Y} = (Y_1, Y_2, \ldots, Y_m)$ is a risk exchange (REX) if

i) $Y_i \geq 0$

ii) $\Sigma Y_i = X$

Let $\text{Premium}[Y_i] = \int Y_i(\omega)\Pi(\omega)d\mu(\omega) = E[Y_i \Pi]$ with $0 < \Pi(\omega) \leq L$

(Π = random variable with values $\Pi(\omega)$) and consider $q = (q_1, q_2, \ldots, q_m)$ as given.

Interpretation: $\Pi(\omega)$ is a "pricing function".

Def. The REX $\underline{Y}$ is called a q-fair risk exchange (q-FAIRREX) if Premium $[Y_i] = q_i$ for all i.
(If it is clear which q we are referring to, then we speak simply of a FAIRREX.)

Let $v_i(x)$ be the disutility of claims to company i. Assume $v_i'(x) > 0$, $v_i''(x) \geq 0$. Then we define

Def. i) The REX $\tilde{\underline{Y}}$ is a Pareto optimal risk exchange (POREX) if there is no other REX $\underline{Y}$ such that for one K

$$\int_{X(\omega) \leq K} v_i(Y_i(\omega))d\mu(\omega) \leq \int_{X(\omega) \leq K} v_i(\tilde{Y}_i(\omega))d\mu(\omega) \quad \text{for all } i$$

$$\int_{X(\omega) \leq K} v_i(Y_i(\omega))d\mu(\omega) < \int_{X(\omega) \leq K} v_i(v_i(\tilde{Y}_i(\omega)))d\mu(\omega) \quad \text{for one } i .$$

ii) The POREX $\tilde{\underline{Y}}$ is called nondegenerate if for all i $Y_i(\omega) > 0$ on a set of positive probability.

Def. A fair POREX is called a FAIRPOREX (q-FAIRPOREX if we are explicitely referring to the particular conditions of fairness).

For later purpose we define R as set of all REX , F_q as set of all q-FAIRREX. In the following we also assume $q_i \geq 0$ and $\sum_i q_i$ = Premium[X], hence F_q is never empty.

3. Theorem 1 (Characterization of a POREX)

$\tilde{Y}$ is a POREX if and only if there exist positive constants $k_1, k_2, \ldots, k_m$ and a positive random variable C such that for almost all ω and all $i = 1,2, \ldots, n$

$$k_i v_i'\left(\tilde{Y}_i(\omega)\right) \geq C(\omega)$$

$$k_i v_i'\left(\tilde{Y}_i(\omega)\right) = C(\omega) \quad \text{if} \quad \tilde{Y}_i(\omega) > 0$$

3.1. We shall need the following

Finite Dimensional Kuhn-Tucker Theorem

For convex differentiable $v_i(x)$ the following statements imply each other.

$$\boxed{\begin{array}{l} \sum_i k_i v_i(\tilde{x}_i) = \min \\ \text{under } 1)\ x_i \geq 0 \\ \qquad\quad 2)\ \sum x_i = x \end{array}} \iff \boxed{\begin{array}{l} k_i v_i'(\tilde{x}_i) \geq C \\ k_i v_i'(\tilde{x}_i) = C \quad \text{if} \quad \tilde{x}_i > 0 \\ \text{with } 1)\ \tilde{x}_i \geq 0 \\ \qquad\quad 2)\ \sum \tilde{x}_i = x \end{array}}$$

3.2. Proof of Theorem 1

$\boxed{\underline{\tilde{Y}} \text{ POREX}}$

i) $\Updownarrow$

i) $\Uparrow$ immediate

$\Downarrow$ from convexity of R and $v_i(x)$ and the fact that $\underline{\tilde{Y}}$ lies on the efficient boundary. $k_i>0$ for all i follows from nondegeneracy, but also in the degenerate case one can choose $k_i > 0$.

There exist positive constants $k_1, k_2, \ldots, k_m$ such that

$$\sum_i k_i \int_{X(\omega)\leq K} v_i(\tilde{Y}_i(\omega))\,d\mu(\omega) \leq \sum k_i \int_{X(\omega)\leq K} v_i(Y_i(\omega))\,d\mu(\omega)$$

for all K and all REX $\underline{Y}$.

ii) $\Updownarrow$

ii) $\Uparrow$ immediate

$\Downarrow$ by modification argument

$$\sum_i k_i v_i(\tilde{Y}_i(\omega)) \leq \sum_i k_i v_i(Y_i(\omega))$$

for almost all ω and for all $\underline{Y}$ with

1) $Y_i(\omega) \geq 0$

2) $\sum_i Y_i(\omega) = X(\omega)$

iii) $\Updownarrow$

iii) finite dimensional Kuhn-Tucker Theorem

$$k_i v_i'(Y_i(\omega)) \geq C(\omega)$$

$$k_i v_i'(\tilde{Y}_i(\omega)) = C(\omega) \quad \text{if} \quad \tilde{Y}_i(\omega) > 0$$

for almost all ω.

4. Theorem 2

> Suppose all $v_i'(0) > 0$ and $X \leq K_0$ then
>
> i) any nondegenerate q-FAIRREX $\tilde{\underline{Y}}$ which minimizes $\phi(\underline{Y})$ among all q-FAIRREX is a nondegenerate q-FAIRPOREX and vice versa.
>
> ii) The nondegenerate q-FAIRPOREX is unique, provided for at least one i, $v_i''(x) > 0$.

4.1. Define for any REX $\underline{Y} \in R$

$$\phi(\underline{Y}) = \sum_{i=1}^{m} \int \psi_i\big(Y_i(\omega)\big)\Pi(\omega)\,d\mu(\omega) = \sum_{i=1}^{m} E[\psi_i(Y_i)\Pi] \; ,$$

where $\psi_i(z) = \int_0^z \log v_i'(x)\,dx$.

Because X is bounded $\phi(\underline{Y})$ is finite for all REX. It is easily checked that for any convex linear combination of REX $\underline{Y}^{(1)}$ and REX $\underline{Y}^{(2)}$ we have

$$\phi\big(\lambda\underline{Y}^{(1)} + (1-\lambda)\underline{Y}^{(2)}\big) \leq \lambda\phi(\underline{Y}^{(1)}) + (1-\lambda)\phi(\underline{Y}^{(2)})$$

(follows from convexity of $\psi_i(z)$).

Hence $\phi(\underline{Y})$ is convex on R (strictly convex if at least for one i $v_i''(x) > 0$)

4.2. Consider the set F_q of all q-FAIRREX and show existence of a q-FAIRREX minimizing ϕ on F_q. This is seen as follows:
On R consider the weak* topology i.e. $\underline{Y}^{(n)} \to \underline{Y}$ if $E[Y_i^{(n)}\rho] \to E[Y_i\rho]$ for all i and all $\rho \in L_\infty$. In this topology F_q is closed. As $0 \leq Y_i \leq X$, all REX components are uniformly integrable, hence by the compactness criterion of Dunford-Pettis (see [6], page 43) F_q is compact.

Define $\underline{Y} \in F_q$, $\sigma = \inf \phi(Y)$ (possibly $\sigma = -\infty$) .
Then we know that there is a sequence $\underline{Y}^{(n)}$ which converges weak*ly to $\underline{Y} \in F_q$ with

$$\phi(\underline{Y}^{(n)}) \to \sigma \ .$$

But $\quad \psi_i(Y_i^{(n)}) \geq \psi_i(Y_i) + \psi_i'(Y_i)(Y_i^{(n)} - Y_i)$.

Hence $\quad \phi(\underline{Y}^{(n)}) \geq \phi(\underline{Y}) + \sum_{i=1}^{m} E[\psi_i'(Y_i)(Y_i^{(n)} - Y_i)\Pi]$.

As $\psi_i'(Y_i)$ is bounded, we have from weak* convergence

$$\sigma \geq \phi(\underline{Y}) \Rightarrow \phi(\underline{Y}) = \sigma$$

q.e.d.

4.3. For the proof of theorem 2 we need the following
Infinite Dimensional Kuhn-Tucker Theorem:

Let ϕ be a convex function defined on a convex set R in a real linear space H . Let $L = (L_1, L_2, \ldots, L_m)$ be a linear function from H to R^m . Assume q in the relative interior of $L(R)$ and that for each $b \in L(R)$ the function ϕ attains a minimum in $L^{-1}(b)$. Then

$\phi(\tilde{\underline{Y}}) = \min$ under 1) $\underline{Y} \in R$ 2) $L(\underline{Y}) = b$	$\Longleftrightarrow$	There exists an m vector $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $\phi(\tilde{\underline{Y}}) - \sum_{j=1}^{m} \lambda_j L_j(\tilde{\underline{Y}}) = \min$ under 1) $\underline{Y} \in R$

4.4. Proof of Theorem 2

First observe that we have $q_i > 0$ for all i in the below q defining the q-FAIRPOREX (If there are $q_i = 0$, the corresponding FAIRREX is degenerate). From this remark and section 4.2 we find that the conditions for the Infinite Kuhn-Tucker theorem are fulfilled. Hence we have

for i)
$$\boxed{\begin{array}{l} \underline{\tilde{Y}} \text{ minimizes } \phi(\underline{Y}) \\ \text{under 1) } \underline{Y} \in R \\ \qquad\quad 2)\ L(\underline{Y}) = q \end{array}}$$

$\Updownarrow$ infinite dimensional Kuhn-Tucker theorem

$$\boxed{\begin{array}{l} \underline{\tilde{Y}} \text{ minimizes } \sum_{j=1}^{m} \left(E[\psi_j(Y_j)\Pi] - \lambda_j E[Y_j \Pi]\right) \\ \text{under 1) } \underline{Y} \in R \end{array}}$$

$\Updownarrow$ same argument as for theorem 1 (observe $\Pi(\omega) > 0$ for all ω !)

$$\boxed{\begin{array}{l} \underline{\tilde{Y}}(\omega) \text{ minimizes } \sum_{j=1}^{m} \left(\psi_j(Y_j(\omega)) - \lambda_j Y_j(\omega)\right) \quad \text{for all } \underline{Y} \\ \text{with 1) } Y_j(\omega) > 0 \\ \qquad\quad 2)\ \sum_j Y_j(\omega) = X(\omega) \\ \text{and for almost all } \omega \ . \end{array}}$$

$\Updownarrow$ finite dimensional Kuhn-Tucker

$$\boxed{\begin{array}{l} k_i v_i'(Y_i(\omega)) > C(\omega) = e^{D(\omega)} > 0 \quad \text{for all } i \\ k_i v_i'(Y_i(\omega)) = C(\omega) \quad \text{if } \tilde{Y}_i(\omega) > 0 \\ \qquad\qquad\qquad\qquad\quad \text{and for almost all } \omega \end{array}}$$

According to theorem 1 this characterizes Pareto optimality.

For ii) If $v_i''(\omega) > 0$ for one i, then $\phi(\underline{Y})$ is strictly convex and hence $\underline{\tilde{Y}}$ is unique.

5. Theorem 3

Now we drop the condition that X is uniformly bounded, replacing it by some weaker assumption:

> i) Suppose $X \in L_1$ and $q_i > 0$ for all i.
> Then there is a (nondegenerate) q-FAIRPOREX $\underline{\tilde{Y}}$
>
> ii) If $\phi(\underline{\tilde{Y}}) < \infty$ then the (nondegenerate) FAIRPOREX is unique.

5.1. Proof for existence

Let $K_n \to \infty$ for $n \to \infty$ and

i) consider the random variables X^{K_n} (truncated at K_n) and for each n let $\underline{Y}^{(n)}$ be the unique q-FAIRPOREX on X^{K_n} guaranteed by theorem 2.

Impose the weak* convergence on R i.e. define

$\underline{Y}^{(n)} \to \underline{Y}$ if $E[\rho \cdot Y_i^{(n)}] \to E[\rho \cdot Y_i]$ for all i and all $\rho \in L_\infty$.

As in 4.2, we conclude that F_q is compact, hence at least a subsequence of $\underline{Y}^{(n)}$ converges to some $\underline{\tilde{Y}} \in F_q$. Hence it follows that $\underline{\tilde{Y}}$ is a q-FAIRREX.

ii) We show that $\underline{\tilde{Y}}$ is also Pareto optimal:

If not, there exist $\underline{Y}$ and K such that

$$\text{(A)} \quad \int_{X(\omega)\leq K} v_i\big(Y_i(\omega)\big)d\mu(\omega) \leq \int_{X(\omega)\leq K} v_i\big(\tilde{Y}_i(\omega)\big)d\mu(\omega) \quad \text{for all } i$$

$$\int_{X(\omega)\leq K} v_i\big(Y_i(\omega)\big)d\mu(\omega) < \int_{X(\omega)\leq K} v_i\big(\tilde{Y}_i(\omega)\big)d\mu(\omega) \quad \text{for one } i$$

The fairness condition implies that, for sufficiently large K, $\tilde{Y}_i(\omega) > 0$ with positive probability on $X(\omega) \leq K$. This allows us to deduce from (A) the stronger condition

$$\text{(B)} \quad \int_{X(\omega)\leq K} v_i\big(Y_i(\omega)\big)d\mu(\omega) < \int_{X(\omega)\leq K} v_i\big(\tilde{Y}_i(\omega)\big)d\mu(\omega) + \varepsilon$$

for all i and sufficiently small $\varepsilon > 0$.

But

$$\int_{X(\omega)\leq K} v_i\big(\tilde{Y}_i(\omega)\big)d\mu(\omega) + \int_{X(\omega)\leq K} v_i'\big(\tilde{Y}_i(\omega)\big)[Y_i^{(n)}(\omega) - \tilde{Y}_i(\omega)]d\mu(\omega) \leq$$

$$\leq \int_{X(\omega)\leq K} v_i\big(Y_i^{(n)}(\omega)\big)d\mu(\omega)$$

Since the second term on the left side tends to zero by weak* convergence, (B) contradicts Pareto optimality of $\underline{Y}^{(n)}$ for sufficiently large n .

5.2. Proof for unicity

For any q-FAIRPOREX $\tilde{\underline{Y}}$ the proof of theorem 2 in the upwards direction applies. If $\phi(\tilde{\underline{Y}}) < \infty$ and $v_i''(x) > 0$ for one i , then there can be no other q-FAIRPOREX $\tilde{\tilde{\underline{Y}}}$ different from $\tilde{\underline{Y}}$.

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